IDENTIFICATION OF NONLINEAR, MEMORYLESS SYSTEMS USING CHEBYSHEV NODES

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ABSTRACT

This paper describes an approach for identification of static nonlinearities from input-output measurements. The approach is based on minimax approximation of memoryless nonlinear systems using Chebyshev polynomials. For memoryless nonlinear systems that are finite and continuous with finite derivatives, it is known that the error caused by the Kth order Chebyshev approximation in a specified interval is bounded by a quantity that is proportional to the maximum value of the $(K+1)$th derivative of the input-output relationship and decays exponentially with K. The method of the paper identifies the system by first estimating the system output at the Chebyshev nodes using localized linear model around the nodes, and then solving for the coefficients associated with the Chebyshev polynomials of the first kind.

1. INTRODUCTION

This paper describes the identification of a nonlinear, memoryless system in a minimax framework. Figure 1 shows a generic block diagram of the identification problem. The system model consists

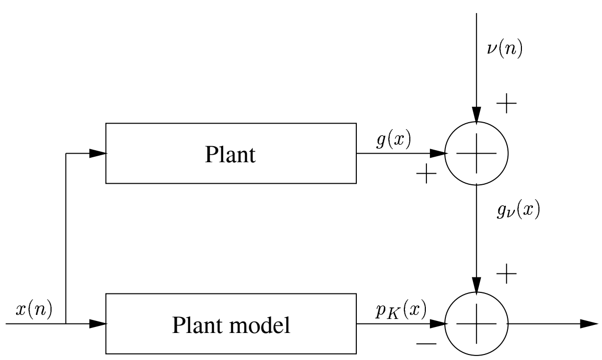

Fig. 1. Block diagram of identification of the nonlinear, memoryless system $g(x)$.

of a Kth order polynomial $p_k(x)$, where x is the input to the system. The underlying plant $g(x)$ is a continuous, finite and memoryless nonlinear function with finite derivatives. The output of the plant is corrupted by additive noise $\nu(n)$, where n denotes a time instant. Polynomials are popular in nonlinear function identification because they can approximate a large number of nonlinearities with good accuracy. However not all polynomials behave equally well. Orthogonal polynomials have several advantages over non-orthogonal polynomials in this problem. For example, the Kth order orthogonal polynomial contributes to only the Kth order distortion term, implying that it is possible to build system models in an order-sequential manner. In general, models based on orthogonal polynomials exhibit better stability in finite precision computations.

In this paper we present an approach for identifying memoryless nonlinear systems using Chebyshev polynomials of the first kind from input-output measurements. Using Chebyshev polynomials has the advantage that they provide the best approximation in the minimax sense to arbitrary, continuous nonlinear functions with finite derivatives in any given finite intervals.

The rest of the paper is organised as follows. The next section contains an overview of the minimax approximation of nonlinear functions. Application of this theory to system identification from input-output measurements is described in Section 3. Experimental results evaluating the capabilities of our approach in a nonlinear system identification problem as well as an equalization problem are given in Section 4. Concluding remarks are provided in Section 5.

2. MINIMAX APPROXIMATION OF NONLINEAR FUNCTIONS

Let $g(x)$ be a continuous function with $(K+1)$ finite derivatives on an interval $[-1,1]$ and let $p_K(x)$ represent an Kth order polynomial approximation for $g(x)$. It is well known that the largest error associated with the best Kth order polynomial approximation for $g(x)$ in the minimax sense is given by

$$E_{max} = \frac{\max\limits_{|x|\leq 1} \left| g^{(K+1)}(x) \right|}{(K+1)!} \cdot \frac{1}{2^K}, \tag{1}$$

where $g^{(K+1)}(x)$ represents the $(K+1)$th order derivative of $g(x)$ [1]. The best approximation is known to be equal to $g(x)$ at $K+1$ points, so called Chebyshev nodes, defined by

$$x_i = \cos\left(\frac{2i+1}{2K+2}\pi\right), \qquad i = 0, 1, \ldots, K \tag{2}$$

and is given by

$$p_K(x) = \sum_{j=0}^{K} C_j T_j(x), \tag{3}$$

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where $T_j(x)$ represent the jth order Chebyshev polynomials of the first kind. The orthogonal polynomials $T_j(x)$ are constructed using the recursion [1]:

$$T_0(x) = 1 \tag{4}$$
$$T_1(x) = x \tag{5}$$

and

$$T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x), \qquad j \geq 1. \tag{6}$$

In (3), the coefficients C_j associated with $T_j(x)$ can be evaluated as

$$C_j = \frac{\xi}{K+1} \sum_{i=0}^{K} g(x_i) T_j(x_i), \tag{7}$$

where

$$\xi = \left\{ \begin{array}{lll} 2, & \text{for} & j \neq 0 \\ 1, & \text{for} & j = 0 \end{array} \right. . \tag{8}$$

Even though the above results apply to $x \in [-1, 1]$, they can be extended to an arbitrary interval $[a, b]$ using an appropriate transformation of the variable x. A short proof of the above results is given in the Appendix.

3. NONLINEAR SYSTEM IDENTIFICATION FROM INPUT-OUTPUT MEASUREMENTS

The fundamental theory described in the previous section requires perfect knowledge of the input-output relationship of the memoryless nonlinearity at the Chebyshev nodes. When system identification is performed from (possibly noisy) input-output data, the exact information about the output values corresponding to input values associated with the Chebyshev nodes are rarely available. Consequently our approach involves finding an approximate model for the input-output relationship in a small neighborhood of each Chebyshev node, and using the output value derived from this highly localized model in the Chebyshev approximation problem. In the experiments presented later in this paper, the local models were linear, and the dynamic range of the input signals for each model corresponded to an interval of length 0.004 surrounding each node. In addition to providing the output values needed by the approximation problem, this approach has the advantage of reducing the impact of measurement noise because of the inherent averaging performed while building the local models.

4. SIMULATION RESULTS

In this section we present the results of two sets of simulation experiments to evaluate the performance capabilities of the method of this paper. The experiments involved identification of an unknown, memoryless nonlinearity and its inverse. The input-output relationship of the unknown plant was

$$g(n) = 0.4267x^5(n) + 0.5733x(n). \tag{9}$$

The input signal was an AWGN bandpass signal with a normalized bandwidth equal to 0.2 and normalized center frequency equal to 0.49. We used a 7th order polynomial model to perform the identification. In order to obtain the value of the nonlinearity at the nodes x_i, we used a linear least-squares approximation of the nonlinearity at the node. We used the input-output sample pairs associated with all input samples in the range $[x_i - 0.002, x_i + 0.002]$. If this interval did not provide at least 15 points, the range was appropriately enlarged to include a sufficient number of points to perform this highly localized linearization.

Table 1. Statistics of the parameter estimates in the system identification example.

	Coefficients		
Coeff.	**True**	**Mean**	**Var.**$[10^{-6}]$
C_0	0	-0.00003554982434	2.08022262
C_1	0.8399875	0.83977374137880	5.64273039
C_2	0	0.00009637033150	2.77009029
C_3	0.13334375	0.13362840020765	3.79980302
C_4	0	0.00008471339529	3.87939764
C_5	0.02666875	0.02719141657395	3.55426233
C_6	0	0.00006123399498	5.15691189
C_7	0	0.00026341756657	1.27910296

We performed several simulations in which the output of the plant (9) was corrupted with additive white Gaussian noise with zero mean value and variance such that the output SNR varied from 0 dB to 50 dB in steps of 5 dB. We also performed an experiment with no additive noise. We conducted 50 independent experiments for each noise level using 50,000 input samples each.

Figure 2 shows the plots of the maximum identification error over all experiments, the mean value of the maximum error over the 50 runs and the variance of the maximum error as a function of the output SNR. The no noise case is represented by `Inf` in the graph. We can see from the figure that the maximum error and its mean value as well as variance decreased exponentially with increasing SNR. The larger than zero error for the infinite SNR case is due to the errors caused by the localized linearization the method performs to estimate the output signals at the Chebyshev nodes. Table 1 shows the ideal coefficients of the orthogonal system, the mean values and variances of the coefficients obtained from the identification with 20 dB output additive noise. Even though the input-output relationship of the unknown plant was a polynomial

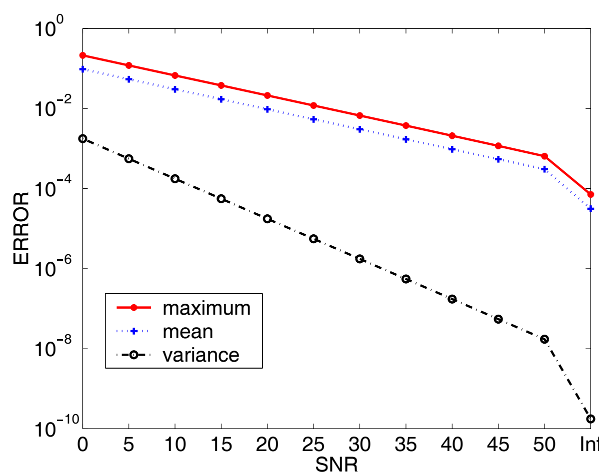

Fig. 2. Maximum absolute error over 50 experiments *vs.* noise levels (solid dots), average of maximum absolute error over all runs for each noise level (crosses) and variances of maximal absolute errors (circles).

of the fifth order, we chose to model it with a seventh order polynomial to demonstrate robustness of the approach to model mismatch. Figure 3 displays the power spectrum of the input signal, the output signal and the difference signal between the output of the identified system model and the noise free output of the plant for the cases when the output SNR was 20 dB and infinity. From the figures, we

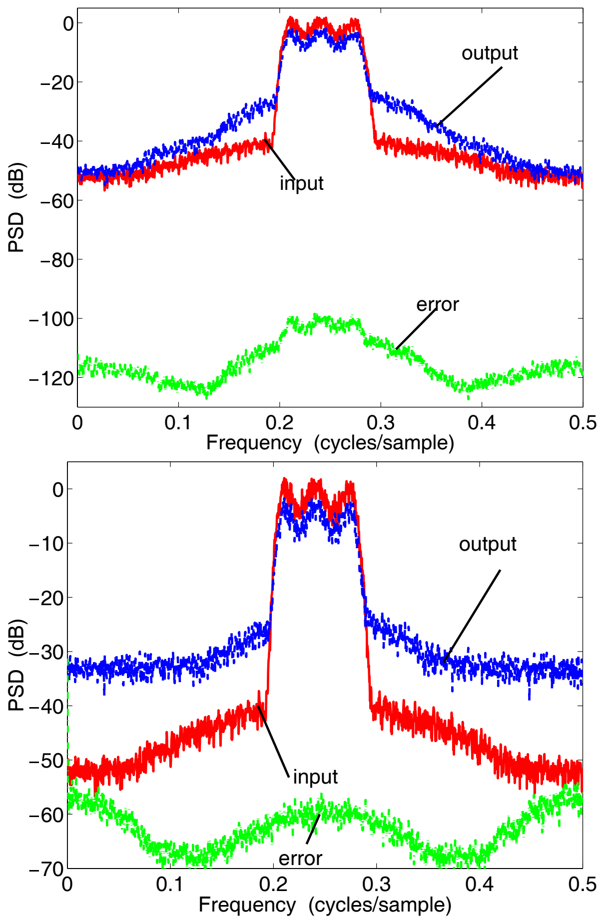

Fig. 3. Power spectrum of the input signal, the output signal and the error signal. Top: No noise is present. Bottom: SNR = 20 dB.

can see that the spectrum of the estimation error is relatively small even in the case when measurement noise is present at the output. These results indicate that even though the theory was based on exact knowledge of the input-output relationship, our approach is robust to the presence of noise in the measurements.

A second second set of simulations were conducted to evaluate the capability of our approach in a nonlinear equalization problem. A block diagram describing this problem is shown in Figure 4. The nonlinear system to be equalized was the same as the one given in (9). The measurement noise at the output of the forward system corresponded to an SNR value of 20 dB. The equalizer was designed using a 21^{st} order filter. The results of the simulations are presented in Figure 5, which shows the power spectra of the input signal to the forward system, the noisy output of the forward system and the estimation error signal. The estimation error signal was obtained as the difference between the input signal and the output of a cascade of the forward system and its equalizer. Since the error

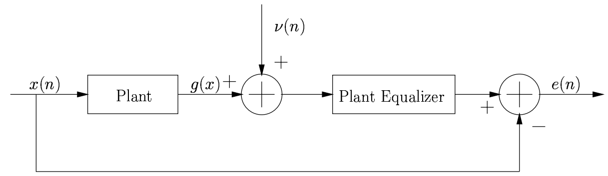

Fig. 4. Nonlinear equalization.

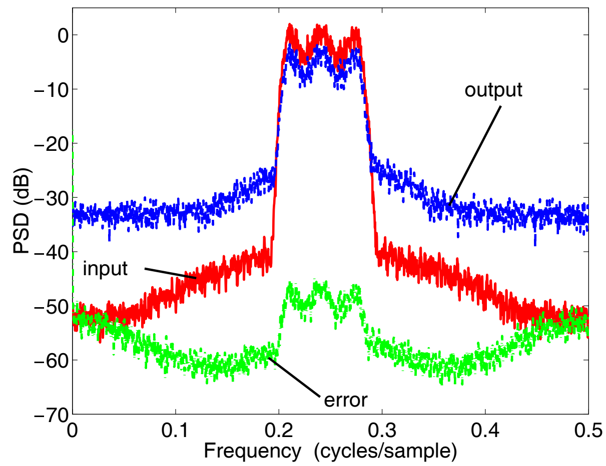

Fig. 5. Power spectrum of the input signal, the output signal and the error signal in a nonlinear equalization problem.

displayed in the figure is relatively low, we infer that the method works well in equalization problems even in the presence of measurement noise. Another advantage of this inversion approach is that it does not require knowledge of the parameters of the forward system as the approach in [2] needs. Furthermore, the performance of the method does not depend on the properties of the input signal.

5. CONCLUSIONS

This paper described a minimax approach to identification of nonlinear memoryless systems from input-output measurements. The performance analysis done through simulations indicate that the method is robust to the presence of measurement noise in the signals. Extension of the work to systems with memory and evaluation of its use in a variety of applications are currently underway.

6. APPENDIX

Let ξ be any (fixed) point other than a node x_i and define

$$w(\xi) = \prod_{i=0}^{K} (\xi - x_i), \tag{10}$$

where x_i is defined in (2). We further define

$$\phi(x) = g(x) - p_K(x) - \lambda w(x), \tag{11}$$

where $g \in C^{K+1}[a,b]$ and λ is a real number such that $\phi(\xi) = 0$ in $[-1, 1]$. We use a polynomial $p_K(x)$, where subscript K denotes the order of the polynomial, to approximate the estimated nonlinearity in the interval $[-1, 1]$.

The $(K + 1)$th derivative of (11) is expressed as

$$\begin{aligned} \phi^{(K+1)}(x) &= g^{(K+1)}(x) - p_K^{(K+1)}(x) - \lambda w^{(K+1)}(x) \\ &= g^{(K+1)}(x) - (K+1)! \cdot \lambda. \end{aligned} \tag{12}$$

Because $p_K(x)$ is a Kth order polynomial, its $(K+1)$th derivative is equal to zero. Furthermore, ϕ is equal to zero at $K + 2$ points $\xi, x_0, x_1, \ldots, x_K$, and the first derivative of ϕ has at least $K + 1$ distinct zeros in the interval $(-1, 1)$. Similarly, $\phi^{(2)}$ has at least K distinct zeros in $(-1, 1)$, whereas $(K + 1)$th derivative has at least one zero η which turns (12) into

$$\begin{aligned} \phi^{(K+1)}(\eta) &= 0 \\ &= g^{(K+1)}(\eta) - (K+1)! \cdot \lambda. \end{aligned} \tag{13}$$

Substituting for λ from (11) and noting that $\phi(\xi) = 0$ we get

$$g^{(K+1)}(\eta) - (K+1)! \cdot \frac{g(\xi) - p_K(\xi)}{w(\xi)} = 0. \tag{14}$$

Using (10) and (14) gives

$$g(\xi) - p_K(\xi) = \frac{g^{(K+1)}(\eta)}{(K+1)!} \prod_{i=0}^{K} (\xi - x_i). \tag{15}$$

Since ξ is arbitrarily chosen in $[-1, 1]$, (15) is applicable for all $\xi \in [-1, 1]$. Therefore, in what follows, we replace ξ with x.

The polynomial approximation error is found from (15) as [3]

$$E_K(x) = g(x) - p_K(x) = \frac{g^{(K+1)}(\eta)}{(K+1)!} \prod_{i=0}^{K} (x - x_i). \tag{16}$$

We note that we do not know precisely the value of η in (16). However, this information is not required for finding the maximum error $E_{max} = \max_{x \in [-1,1]} |g(x) - p_K(x)|$. In particular,

$$E_{max} \leq \left\{ \frac{\max_{x \in [-1,1]} \left| g^{(K+1)}(\eta) \right|}{(K+1)!} \right\} \left\{ \max_{x \in [-1,1]} \left| \prod_{i=0}^{K} (x - x_i) \right| \right\}. \tag{17}$$

The following theorem from [1] allows us to find the smallest value of $\left\{ \max_{x \in [-1,1]} \left| \prod_{i=0}^{K} (x - x_i) \right| \right\}$. It states that among all polynomials $p_K(x)$, the Chebyshev polynomial of the first kind $T_K(x)$ hugs the x axis in the interval $[-1, 1]$ more closely than any other.

Theorem 1 (Chebyshev polynomial [1]) *Let $p_K(x)$ be any polynomial of Kth degree with leading coefficient one. Then*

$$\max \left| \frac{T_K(x)}{2^{K-1}} \right| \leq \max |p_K(x)| \tag{18}$$

in the interval $-1 \leq x \leq 1$.

Proof We normalize the Chebyshev polynomial $T_K(x)$ with its leading coefficient 2^{K-1} to define

$$T_K^*(x) = 2^{-K+1} T_K(x) \tag{19}$$

so that the leading coefficient of $T_K^*(x)$ is unity. The following explicit expression involving trigonometric functions is known [4, 1] for the Chebyshev polynomials of the first kind:

$$T_K(\cos \varphi) = \cos K\varphi. \tag{20}$$

Let $x = cos\varphi$, and use (20) with (19), to get

$$T_K^*(x) = 2^{-K+1} \cos\left(K \cos^{-1}(x)\right). \tag{21}$$

The extreme values of $T_K^*(x)$ are given at

$$\zeta_i = \cos(i\pi/K), \qquad i = 0, 1, \ldots, K \tag{22}$$

where

$$T_K^*(\zeta_i) = 2^{-K+1} \cos i\pi = (-1)^i 2^{-K+1}. \tag{23}$$

This result says that the maximum deviations of $T_K^*(x)$ from the x axis are

$$T_K^*(x) = \pm \left(\frac{1}{2}\right)^{K-1}. \tag{24}$$

Since both polynomials $p_K(x)$ and $T_K^*(x)$ have the leading coefficients equal to one, the difference of the two polynomials

$$p_K(x) - T_K^*(x) \tag{25}$$

is a polynomial of at most $(K - 1)$th degree. If we assume that $\max |T_K^*(x)| > \max |p_K(x)|$ at any point $x \in [-1, 1]$, the polynomial $p_K(x) - T_K^*(x)$ would due to (23) change the sign at least K times, which means that the polynomial $p_K(x) - T_K^*(x)$ has at least K roots. This is not possible for a polynomial of $(K - 1)$th order, which proves (18).

Q.E.D.

Note that $w(x)$ with roots $x_i, \ i = 0, 1, ..., K$ is indeed the $(K+1)$th order Chebyshev polynomial. Therefore we can combine (16) and (24) to get the bound on the maximum error as

$$E_{max} \leq \frac{\max_{|\eta| \leq 1} \left| g^{(K+1)}(\eta) \right|}{(K+1)!} \cdot \frac{1}{2^K}. \tag{26}$$

7. REFERENCES

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